UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

An Introduction to the Fractional Calculus: Theories and Applications

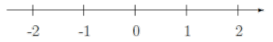
By:

Ali Ahmadian
Postdoctoral Research Fellow
Department of Mathematics

BERILMU BERBAKTI
WITH KNOWLEDGE WE SERVE
www.upm.edu.my

An Overview:

... from integer to non-integer ...



$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n}$$

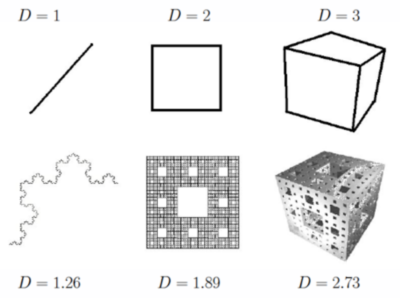
$$x^n = e^{n \ln x}$$

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n,$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad x > 0,$$

$$\Gamma(n+1) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n!$$

... from integer to non-integer ...



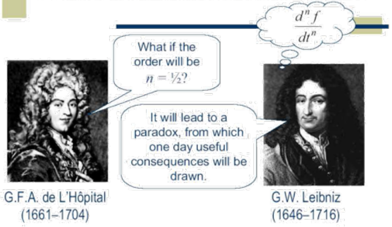
Interpolation of operations

$$f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \frac{d^3 f}{dt^3}, \quad \ldots$$

$$f, \quad \int f(t)dt, \quad \int dt \int f(t)dt, \quad \int dt \int dt \int f(t)dt, \quad \ldots$$

$$\ldots, \quad \frac{d^{-2} f}{dt^{-2}}, \quad \frac{d^{-1} f}{dt^{-1}}, \quad f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \ldots$$

Fractional Calculus was born in 1695



Integer-Order Calculus

Fractional-Order Calculus

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

The advantages of fractional order models

They depict the real world objects more accurately;
Variable stability region;
etc...

G.W. Scott Blair (1950)

"We may express our concepts in Newtonian terms if we find this convenient but, if we do so, we must realize that we have made a translation into a language which is foreign to the system which we are studying."

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Mathematics Books

- The first book dedicated specifically to the theory of fractional calculus

  K.B. Oldham, J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order (Academic Press, 1974).

- Two remarkably comprehensive encyclopedic-type monographs:

  S.G. Samko, A.A. Kilbas, O.I. Marichev, Integrals and Derivatives of Fractional Order and Applications} (Nauka i Tehnika, Minsk, 1987); Fractional Integrals and Derivatives Theory and Applications (Gordon and Breach, 1993).

  A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations (Elsevier, 2006).

- I. Podlubny, Fractional Differential Equations (Academic Press, 1999).

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Special Journals

- "Journal of Fractional Calculus";
- "Fractional Calculus and Applied Analysis";
- "Fractional Dynamic Systems";
- "Communications in Fractional Calculus".

What is fractional calculus?

- Fractional calculus is the study of $\frac{d^q}{dx^q}(f(x))$ for arbitrary real or complex values of q.
- The term 'fractional' is a misnomer. q need not necessarily be a fraction (rational number).
- If $q > 0$ we have a *fractional derivative* of order q.
- If $q < 0$ we have a *fractional integral* of order $-q$.

- All of us are familiar with *normal* derivatives and integrals, like, $\frac{df}{dt}$, $\frac{d^2f}{dt^2}$, $\int_0^t f(u)du$.
- We have first-order, second-order derivatives, or first integral, double integral, of a function.
- Now we wish to have half-order, πth-order, or . . . . . . (3-6i)th-order derivative of a function.
- So, Fractional calculus $\Rightarrow$ derivatives and integrals of arbitrary real, or complex order

Compare with the meaning of a^n.

- The situation is similar to the problem of defining, and giving a meaning and an interpretation to, a^n in the case where n is not a positive integer.
- If n is a positive integer, then a^n is the result of multiplying a by itself n times. If n is not a positive integer, can we visualise a^n as multiplication of a by itself n times?
- Is $a^{1/2}$ the result of multiplying a by itself $\frac{1}{2}$ times?

So what?? Answer these questions....

- Does it make any sense? Or is just a mathematical fantasy? Define it.
- Tell me how to calculate the 1/2-order derivative of $f(t) = t$.
- *This seems to be a recent stuff.* How old is it?
- Does it have any physical interpretation/geometrical meaning?
- Why study it? How is it important in engineering? What's the deal?
- How much serious is the research community about it?
- What are its applications?
- So should we discard the integer-order derivatives?

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Short History of Fractional Calculus

- As old as normal, conventional, integer-order calculus.
- Born in **1695**!!
- In a letter correspondence, l'Hôpital asked Leibniz: "What if the order of the derivative is 1/2"?
- To which Leibniz replied in a prophetical way, "Thus it follows that will be equal to $x^2\sqrt[2]{dx : x}$, an apparent paradox, from which one day useful consequences will be drawn."
- This letter of Leibniz was dated 30th September, 1695. So **30th September** is considered as the **birthday** of fractional calculus.

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

- **Fifteen** Books.
- **Two** dedicated international journals.
- **First** international conference on "Fractional Calculus and its Applications" in June, 1974 in US.
- Special international conference conducted (first was in 2004) by the International Federation of Automatic Control (IFAC) every two years: **Fractional Differentiation and its Applications**.
- More than 5000 papers published on the single topic of modeling of complex systems by fractional differential equations.

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

- Rigorous mathematical theory has been developed.
- Integer-order calculus is the special case.
- Geometrical interpretation or physical meaning exists. But not as straight forward as for the integer-order derivatives.
- There are more that FIFTEEN definitions of fractional derivative operator.

G. W. Leibniz (1695–1697)

In the letters to J. Wallis and J. Bernulli (in 1697) Leibniz mentioned the possible approach to fractional-order differentiation in that sense, that for non-integer values of n the definition could be the following:

$$\frac{d^n e^{mx}}{dx^n} = m^n e^{mx},$$

L. Euler (1730)

$$\frac{d^n x^m}{dx^n} = m(m-1)\ldots(m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)\ldots(m-n+1)\,\Gamma(m-n+1)$$

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}.$$

Euler suggested to use this relationship also for negative or non-integer (rational) values of n. Taking $m = 1$ and $n = \frac{1}{2}$, Euler obtained:

$$\frac{d^{1/2}x}{dx^{1/2}} = \sqrt{\frac{4x}{\pi}} \qquad \left(= \frac{2}{\sqrt{\pi}}x^{1/2}\right)$$

J. B. J. Fourier (1820–1822)

The first step to generalization of the notion of differentiation for **arbitrary functions** was done by J. B. J. Fourier (*Théorie Analytique de la Chaleur*, Didot, Paris, 1822; pp. 499–508).

After introducing his famous formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz)dp,$$

Fourier made a remark that

$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz + n\frac{\pi}{2})dp,$$

and this relationship could serve as a definition of the n-th order derivative for non-integer n.

N. H. Abel (1823–1826)

N. H. Abel: Solution de quelques problèmes à l'aide d'intégrales définies (1823). *Œuvres complètes de Niels Henrik Abel*, vol. 1, Grondahl, Christiania, 1881, pp. 11–18.

In fact, Abel solved the equation

$$\int_0^x \frac{s'(\eta)d\eta}{(x-\eta)^\alpha} = \psi(x),$$

for an arbitrary α (and not just for $\alpha = \frac{1}{2}$):

$$s(x) = \frac{\sin(\pi\alpha)}{\pi} x^\alpha \int_0^1 \frac{\psi(xt)dt}{(1-t)^{1-\alpha}}.$$

After that, Abel *expressed* the obtained solution with the help of an integral of order α:

$$s(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{-\alpha}\psi(x)}{dx^{-\alpha}}$$

J. Liouville (1832–1855)

Three approaches:

I. Following Leibniz:

$$\frac{d^m e^{ax}}{dx^n} = a^m e^{ax},$$

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x},$$

$$\frac{d^\nu f(x)}{dx^\nu} = \sum_{n=0}^{\infty} c_n a_n^\nu e^{a_n x}$$

J. Liouville (1832–1855)

Three approaches:

II. *Integrals* of non-integer order:

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{(-1)^\mu\Gamma(\mu)} \int_0^\infty \Phi(x+\alpha)\alpha^{\mu-1}d\alpha$$

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{\Gamma(\mu)} \int_0^\infty \Phi(x-\alpha)\alpha^{\mu-1}d\alpha$$

or (after the substitution $\tau = x+\alpha$, $\tau = x-\alpha$)

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{(-1)^\mu\Gamma(\mu)} \int_x^\infty (\tau-x)^{\mu-1}\Phi(\tau)d\tau$$

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x (x-\tau)^{\mu-1}\Phi(\tau)d\tau.$$

J. Liouville (1832–1855)

Three approaches:

III. *Derivatives* of non-integer order:

$$\frac{d^\mu F(x)}{dx^\mu} = \frac{(-1)^\mu}{h^\mu}\Big(F(x) - \frac{\mu}{1}F(x+h) + \frac{\mu(\mu-1)}{1\cdot 2}F(x+2h) - \dots\Big)$$

$$\frac{d^\mu F(x)}{dx^\mu} = \frac{1}{h^\mu}\Big(F(x) - \frac{\mu}{1}F(x-h) + \frac{\mu(\mu-1)}{1\cdot 2}F(x-2h) - \dots\Big).$$

(Equality is in the sense $\lim_{h\to 0}$).

Liouville was the first, who realized the possibility of consid-ereation of *left-sided* and *right-sided* fractional integrals and derivatives.

G. F. B. Riemann (1847; 1876)

Riemann used a generalization of the Taylor series for obtaining a formula for fractional-order integration, which is given below in contemporary notation:

$$D^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)dt + \psi(t)$$

Riemann introduced an arbitrary (complementary) function $\psi(x)$ because he did not fixed the lower bound of integration c – a disadvantage, which cannot be removed in the framework of his approach.

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

N. Ya. Sonin (1869)
A. V. Letnikov (1872)
H. Laurent (1884)
N. Nekrasov (1888)
K. Nishimoto (1987–)

Cauchy's formula:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} dt$$

For non-integer $n = \nu$ a branch point of the function $(t-z)^{-\nu-1}$ appears instead of a pole:

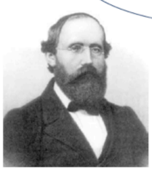
$$D^{\nu} f(z) = \frac{\Gamma(\nu+1)}{2\pi i} \int_c^{x^+} \frac{f(t)}{(t-z)^{\nu+1}} dt$$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Riemann–Liouville definition

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}}$$

$$(n-1 \le \alpha < n)$$



G.F.B. Riemann
(1826–1866)

J. Liouville
(1809–1882)

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

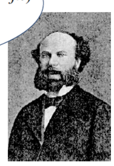
Grünwald–Letnikov definition

$${}_aD_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh)$$

$[x]$ – integer part of x

A.K. Grünwald

A.V. Letnikov



UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Some other definitions

M. Caputo (1967):

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)\,d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 \le \alpha < n) $$

K.S. Miller, B. Ross (1993):

$$ D^{\bar{\alpha}} f(t) = D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_n} f(t), \quad \bar{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) $$

„Left " and „right" fractional derivatives

"Left-sided": $$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}} $$

"Right-sided": $$ {}_tD_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b \frac{f(\tau)\,d\tau}{(\tau-t)^{\alpha-n+1}} $$

a — t — b

${}_aD_t^{\alpha} f(t)$: „past" of $f(t)$

${}_tD_b^{\alpha} f(t)$: „future" of $f(t)$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Approximation

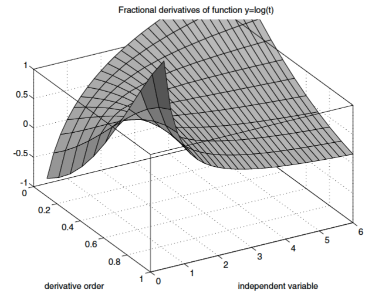
On equidistant mesh with step h the following approximation using *finite differences of fractional order,* which comes from the Grünwald–Letnikov definition:

$$ {}_aD_t^{\alpha} f(t) \approx \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh), $$

$[x]$- integer part of x

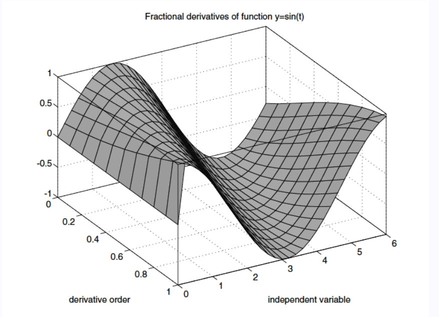
UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Example: $\ln(t)$



UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Example: $\sin(t)$



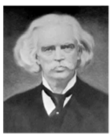
G. M. Mittag-Leffler

Professor Donald E. Knuth, creator of TeX:
"As far as the spacing in mathematics is concerned… I took *Acta Mathematica,* from 1910 approximately; this was a journal in Sweden ... Mittag-Leffler was the editor, and his wife was very rich, and they had the highest budget for making quality mathematics printing. So the typography was especially good in *Acta Mathematica.*"

(Questions and Answers with Prof. Donald E. Knuth, Charles University, Prague, March 1996)

The Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$



One of the most common - and unfounded - reasons as to why Nobel decided against a Nobel prize in math is that [a woman he proposed to/his wife/his mistress] [rejected him because of/cheated him with] a famous mathematician. Gosta Mittag-Leffler is often claimed to be the guilty party. There is no historical evidence to support the story.

In 1882 he founded the Acta Mathematica, which a century later is still one of the world's leading mathematical journals. He persuaded King Oscar II to endow prize competitions and honor various distinguished mathematicians all over Europe. Hermite, Bertrand, Weierstrass, and Poincare were among those honored by the King. (http://db.uwaterloo.ca/~alopez-o/math-faq/node50.html)

Mittag-Leffler function: definition

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \quad \beta > 0)$$

$$E_{1,1}(z) = e^z,$$

$$E_{2,1}(z^2) = \cosh(z), \quad E_{2,2}(z^2) = \frac{\sinh(z)}{z}.$$

$$E_{1/2,1}(z) = e^{z^2}\operatorname{erfc}(-z);$$

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt.$$

Mittag-Leffler function: most important properties

Laplace transform of the M-L function:

$$\int_0^\infty e^{-st} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm a t^\alpha)\,dt = \frac{k!\, s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}},$$

$$(Re(s) > |a|^{1/\alpha}).$$

Differentiation of fractional order:

$${}_0D_t^\gamma \left(t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\lambda t^\alpha)\right) = t^{\alpha k+\beta-\gamma-1} E^{(k)}_{\alpha,\beta-\gamma}(\lambda t^\alpha)$$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad (\alpha > 0, \quad \beta > 0)$$

Humbert, P. and Agarwal, R.P. (1953). Sur la fonction de Mittag-Leffler et quelques-unes de ses generalisations, Bull. Sci. Math. (Ser. II) 77, 180-185.

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z;$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)} = \frac{e^z-1}{z}$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2}\sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)} = \frac{e^z-1-z}{z^2}$$

and in general $\quad E_{1,m}(z) = \frac{1}{z^{m-1}}\left\{e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!}\right\}$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

The problem

$$f, \quad \frac{df}{dt}, \quad \frac{d^2f}{dt^2}, \quad \frac{d^3f}{dt^3}, \quad \ldots$$

$$f, \quad \int f(t)dt, \quad \int dt \int f(t)dt, \quad \int dt \int dt \int f(t)dt, \quad \ldots$$

$$\ldots, \quad \frac{d^{-2}f}{dt^{-2}}, \quad \frac{d^{-1}f}{dt^{-1}}, \quad f, \quad \frac{df}{dt}, \quad \frac{d^2f}{dt^2}, \quad \ldots$$

How to generalize?
Which notation to use?

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Which properties to preserve?

1. For integer orders must give classical derivatives/integrals
2. Zero order derivative: $\frac{d^0 f(t)}{dt^0} = f(t)$
3. The index law:

$$\frac{d^n}{dt^n}\frac{d^m f(t)}{dt^m} = \frac{d^m}{dt^m}\frac{d^n f(t)}{dt^n} = \frac{d^{n+m} f(t)}{dt^{n+m}}$$

4. Linearity:

$$\frac{d^n}{dt^n}\Big(\lambda f(t) + \mu g(t)\Big) = \lambda\frac{d^n f(t)}{dt^n} + \mu\frac{d^n g(t)}{dt^n}$$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Notation

$f^{(\alpha)}(t), \quad \frac{d^\alpha f(t)}{dt^\alpha}$

Various authors, including M. Caputo
M. Caputo,
Elasticità e Dissipazione.
Zanichelli, Bologna, 1969

$\frac{d^\alpha f(t)}{[d(t-a)]^\alpha}, \quad \frac{d^\alpha f(t)}{[d(b-t)]^\alpha}$

M. M. Dzhrbashyan
(various works in 1950s-1960s)

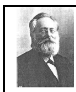
K. B. Oldham and J. Spanier,
The Fractional Calculus.
Academic Press, New York, 1974

${}_aD_t^\alpha f(t), \quad {}_tD_b^\alpha f(t)$

terminals

H. D. Davis,
The Theory of Linear Operators.
Principia Press, Bloomington,
Indiana, 1936

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE



Grünwald-Letnikov approach

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h}$$

backward differences

$$f''(t) = \frac{d^2 f}{dt^2} = \lim_{h\to 0} \frac{f'(t) - f'(t-h)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h}\right\}$$

$$= \lim_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}$$

$$f'''(t) = \frac{d^3 f}{dt^3} = \lim_{h\to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3}$$

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh)$$

binomial coefficients

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Grünwald-Letnikov approach

$$f_h^{(p)}(t) = \frac{1}{h^p} \sum_{r=0}^{n} (-1)^k \binom{p}{r} f(t - rh)$$

p is an arbitrary integer number; n is also integer, as above.

Obviously, for $p \le n$ we have

$$\lim_{h\to 0} f_h^{(p)}(t) = f^{(p)}(t) = \frac{d^p f}{dt^p}$$

What shall we have if p is negative?

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Denote $\left[\begin{matrix} p \\ r \end{matrix}\right] = \frac{p(p+1)\ldots(p+r-1)}{r!}$

Then $\binom{-p}{r} = \frac{-p(-p-1)\ldots(-p-r+1)}{r!} = (-1)^r \left[\begin{matrix} p \\ r \end{matrix}\right]$

and we can write

$$f_h^{(-p)}(t) = h^p \sum_{r=0}^{n} \left[\begin{matrix} p \\ r \end{matrix}\right] f(t - rh)$$

If n is fixed, then

$$\lim_{h\to 0} f_h^{(-p)}(t) = 0$$

Not so interesting...

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Grünwald-Letnikov approach

Left-sided G-L fractional derivative:

$$ {}_aD_t^p f(t) = \lim_{\substack{h \to 0 \\ nh = t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t - rh) $$

Note that this is a generalization
of backward finite differences

Grünwald-Letnikov approach

Letnikov showed that if $f(t)$ has
$m+1$ continuous derivatives in $[a, b]$,
and $m > p - 1$, then (fractional derivative)

$$ {}_aD_t^p f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau) d\tau $$

The smallest possible value for m is such that

$$ m < p < m+1 $$

Grünwald-Letnikov approach

G-L fractional derivative of the power function

$$ f(t) = (t-a)^{\nu} $$

$$ {}_aD_t^p (t-a)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(-p+\nu+1)} (t-a)^{\nu-p} $$

Grünwald-Letnikov approach

Composition with n-th integer order derivatives

Conclusion:

$$ \frac{d^n}{dt^n} \left({}_aD_t^p f(t) \right) = {}_aD_t^p \left(\frac{d^n f(t)}{dt^n} \right) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} $$

$$ \frac{d^n}{dt^n} \left({}_aD_t^p f(t) \right) = {}_aD_t^p \left(\frac{d^n f(t)}{dt^n} \right) = {}_aD_t^{p+n} f(t) $$

only if $f^{(k)}(a) = 0, \quad (k = 0, 1, 2, \ldots, n-1)$

Grünwald-Letnikov approach

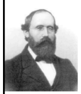
Composition with fractional derivatives

Similarly, it can be shown that

$$ {}_aD_t^q\left({}_aD_t^p f(t)\right) = {}_aD_t^p\left({}_aD_t^q f(t)\right) = {}_aD_t^{p+q} f(t) $$

where $0 \le m < p < m+1$, $0 \le n < q < n+1$

only if $f^{(k)}(a) = 0, \quad (k = 0, 1, \ldots, r-1), \quad r = \max(n, m)$



Riemann-Liouville approach

Manipulation with G-L fractional derivatives defined as a limit of fractional order backward difference is not convenient.

Solution? The Riemann-Liouville fractional derivative:

$$ {}_a\mathbf{D}_t^p f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau, \qquad (m \le p < m+1) $$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Riemann-Liouville approach

Direct link to the G-L fractional derivative

if $f(t)$ has $m+1$ continuous derivatives in $[a,b]$, then repeated integration by parts and differentiation gives immediately:

$$ \begin{aligned} {}_a\mathbf{D}_t^p f(t) &= \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau \\ &= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \\ &\quad + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)d\tau \\ &= {}_aD_t^p f(t), \qquad (m \le p < m+1). \end{aligned} $$

R-L ↗

G-L ↘

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Riemann-Liouville approach

Properties

For the fractional integrals we have:

$$ {}_a\mathbf{D}_t^{-p}\left({}_a\mathbf{D}_t^{-q} f(t)\right) = {}_a\mathbf{D}_t^{-q}\left({}_a\mathbf{D}_t^{-p} f(t)\right) = {}_a\mathbf{D}_t^{-p-q} f(t) $$

This is similar to

$$ \frac{d^m}{dt^m}\left(\frac{d^n f(t)}{dt^n}\right) = \frac{d^n}{dt^n}\left(\frac{d^m f(t)}{dt^m}\right) = \frac{d^{m+n} f(t)}{dt^{m+n}} $$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Riemann-Liouville approach
Fractional derivative of the power function

$$f(t) = (t-a)^{\nu}$$

Recall the definition of the R-L derivative:

$${}_a\mathbf{D}_t^p f(t) = \frac{d^n}{dt^n}\left({}_a\mathbf{D}_t^{-(n-p)} f(t)\right), \qquad (n-1 \le p < n)$$

Evaluate the fractional integral (note p<0):

$$\begin{aligned} {}_aD_t^p (t-a)^{\nu} &= \frac{1}{\Gamma(-p)}(t-a)^{\nu-p}\int_0^1 \xi^{\nu}(1-\xi)^{-p-1}d\xi \\ &= \frac{1}{\Gamma(-p)}B(-p,\nu+1)(t-a)^{\nu-p} \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu-p+1)}(t-a)^{\nu-p}, \quad (p<0,\ \nu>-1) \end{aligned}$$

Riemann-Liouville approach
Fractional derivative of the power function

$$f(t) = (t-a)^{\nu}$$

From the previous formulas we have:

$${}_a\mathbf{D}_t^p\left((t-a)^{\nu}\right) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)}(t-a)^{\nu-p}$$

The only restriction is that $f(t) = (t-a)^{\nu}$ must be integrable, that is $\nu > -1$.

Riemann-Liouville approach
Composition with integer order derivatives

$$\frac{d^n}{dt^n}\left({}_a\mathbf{D}_t^{k-\alpha} f(t)\right) = \frac{1}{\Gamma(\alpha)}\frac{d^{n+k}}{dt^{n+k}}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau = {}_a\mathbf{D}_t^{n+k-\alpha} f(t).$$

$$(0 < \alpha \le 1)$$

Denoting $p = k - \alpha$

$$\frac{d^n}{dt^n}\left({}_a\mathbf{D}_t^p f(t)\right) = {}_a\mathbf{D}_t^{n+p} f(t).$$

Riemann-Liouville approach
Composition with integer order derivatives

Take into account that

$${}_a\mathbf{D}_t^{-n} f^{(n)}(t) = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f^{(n)}(\tau)d\tau = f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)}$$

and

$${}_a\mathbf{D}_t^p g(t) = {}_a\mathbf{D}_t^{p+n}\left({}_a\mathbf{D}_t^{-n} g(t)\right)$$

Then

$$\begin{aligned} {}_a\mathbf{D}_t^p\left(\frac{d^n f(t)}{dt^n}\right) &= {}_a\mathbf{D}_t^{p+n}\left({}_a\mathbf{D}_t^{-n} f^{(n)}(t)\right) \\ &= {}_a\mathbf{D}_t^{p+n}\left(f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)}\right) \\ &= {}_a\mathbf{D}_t^{p+n} f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^{j-p-n}}{\Gamma(1+j-p-n)} \end{aligned}$$

Riemann-Liouville approach
Composition with integer order derivatives

Conclusion: we have

$$\frac{d^n}{dt^n}\left({}_a\mathbf{D}_t^p f(t)\right) = {}_a\mathbf{D}_t^p\left(\frac{d^n f(t)}{dt^n}\right) = {}_a\mathbf{D}_t^{p+n} f(t)$$

only if

$$f^{(k)}(a) = 0, \qquad (k = 0, 1, 2, \ldots, n-1).$$

The same as in case of Grunwald-Letnikov derivatives!

Riemann-Liouville approach
Composition with fractional derivatives

Consider first

$$\begin{aligned}
{}_a\mathbf{D}_t^p\left({}_a\mathbf{D}_t^q f(t)\right) &= \frac{d^m}{dt^m}\left\{{}_a\mathbf{D}_t^{-(m-p)}\left({}_a\mathbf{D}_t^q f(t)\right)\right\} \\
&= \frac{d^m}{dt^m}\left\{{}_a\mathbf{D}_t^{p+q-m} f(t) \right. \\
&\quad \left. - \sum_{j=1}^{n}\left[{}_a\mathbf{D}_t^{q-j} f(t)\right]_{t=a}\frac{(t-a)^{m-p-j}}{\Gamma(1+m-p-j)}\right\} \\
&= {}_a\mathbf{D}_t^{p+q} f(t) - \sum_{j=1}^{n}\left[{}_a\mathbf{D}_t^{q-j} f(t)\right]_{t=a}\frac{(t-a)^{-p-j}}{\Gamma(1-p-j)}.
\end{aligned}$$

Riemann-Liouville approach
Composition with fractional derivatives

Now let us interchange these derivatives (swap p and q, and also m and n):

$${}_a\mathbf{D}_t^q\left({}_a\mathbf{D}_t^p f(t)\right) = {}_a\mathbf{D}_t^{p+q} f(t) - \sum_{j=1}^{m}\left[{}_a\mathbf{D}_t^{p-j} f(t)\right]_{t=a}\frac{(t-a)^{-q-j}}{\Gamma(1-q-j)}.$$

This is different from the previous formula. Therefore, Riemann-Liouville derivatives, in general, do not commute.

Riemann-Liouville approach
Composition with fractional derivatives

... R-L derivatives do not commute. With one exception:

$${}_a\mathbf{D}_t^p\left({}_a\mathbf{D}_t^q f(t)\right) = {}_a\mathbf{D}_t^q\left({}_a\mathbf{D}_t^p f(t)\right) = {}_a\mathbf{D}_t^{p+q} f(t) \qquad (p \neq q)$$

if

$$\left[{}_a\mathbf{D}_t^{p-j} f(t)\right]_{t=a} = 0, \quad (j = 1, 2, \ldots, m),$$

and

$$\left[{}_a\mathbf{D}_t^{q-j} f(t)\right]_{t=a} = 0, \quad (j = 1, 2, \ldots, n).$$

Riemann-Liouville approach
Composition with fractional derivatives

The pervious restrictions on initial values of fractional derivatives are equivalent to:

$$\left[{}_a\mathbf{D}_t^{p-j} f(t)\right]_{t=a} = 0, \quad (j = 1, 2, \ldots, m), \quad \longleftrightarrow \quad f^{(j)}(a) = 0, \quad (j = 0, 1, 2, \ldots, m-1)$$

$$\left[{}_a\mathbf{D}_t^{q-j} f(t)\right]_{t=a} = 0, \quad (j = 1, 2, \ldots, n). \quad \longleftrightarrow \quad f^{(j)}(a) = 0, \quad (j = 0, 1, 2, \ldots, n-1)$$

Riemann-Liouville approach
Composition with fractional derivatives

Conclusion: Riemann-Liouville fractional derivatives commute, that is

$${}_a\mathbf{D}_t^{p}\left({}_a\mathbf{D}_t^{q} f(t)\right) = {}_a\mathbf{D}_t^{q}\left({}_a\mathbf{D}_t^{p} f(t)\right) = {}_a\mathbf{D}_t^{p+q} f(t)$$

if $f^{(j)}(a) = 0, \quad (j = 0, 1, 2, \ldots, r-1),$

$$r = \max(n, m).$$

The same as in case of Grunwald-Letnikov derivatives.

Caputo approach
Why it appeared

Initial conditions for fractional differential equations with Riemann-Liouville derivatives contain

$$\lim_{t \to a} {}_a\mathbf{D}_t^{\alpha-1} f(t) = b_1,$$
$$\lim_{t \to a} {}_a\mathbf{D}_t^{\alpha-2} f(t) = b_2,$$
$$\ldots,$$
$$\lim_{t \to a} {}_a\mathbf{D}_t^{\alpha-n} f(t) = b_n,$$

and / or their combinations.
Troubles with interpretations...

Caputo approach

M. Caputo (1967)

$${}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau) d\tau}{(t - \tau)^{\alpha + 1 - n}}, \quad (n - 1 < \alpha < n).$$

Note the interchange of fractional integration and integer-order differentiation compared to Riemann-Liouville approach.

Caputo approach
Really different from Riemann-Liouville derivative

Riemann-Liouville derivative of a constant A:

$$ {}_0D_t^{\alpha} A = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} $$

Caputo derivative of a constant A:

$$ {}_0^C D_t^{\alpha} A = 0 $$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Caputo approach
Really different from Riemann-Liouville derivative

Another difference - interchange of derivatives:

$$ {}_a^C D_t^{\alpha} \left({}_a^C D_t^{m} f(t) \right) = {}_a^C D_t^{m} \left({}_a^C D_t^{\alpha} f(t) \right) = {}_a^C D_t^{\alpha+m} f(t), $$

$$ f^{(s)}(0) = 0, \quad s = n, n+1, \ldots, m $$

$$ (m = 0, 1, 2, \ldots; n-1 < \alpha < n) $$

$$ {}_a\mathbf{D}_t^{m} \left({}_a\mathbf{D}_t^{\alpha} f(t) \right) = {}_a\mathbf{D}_t^{\alpha} \left({}_a\mathbf{D}_t^{m} f(t) \right) = {}_a\mathbf{D}_t^{\alpha+m} f(t), $$

$$ f^{(s)}(0) = 0, \quad s = 0, 1, 2, \ldots, m $$

$$ (m = 0, 1, 2, \ldots; n-1 < \alpha < n) $$

Caputo: no restrictions on

$$ f^{(s)}(0), \quad (s = 0, 1, \ldots, n-1). $$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Linearity of fractional derivatives

$$ D^p\Big(\lambda f(t) + \mu g(t)\Big) = \lambda\, D^p f(t) + \mu\, D^p g(t) $$

Grunwald-Letnikov derivatives:

$$ \begin{aligned} {}_aD_t^p\Big(\lambda f(t) + \mu g(t)\Big) &= \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} \Big(\lambda f(t-rh) + \mu g(t-rh)\Big) \\ &= \lambda \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) \\ &\quad + \mu \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} g(t-rh) \\ &= \lambda\, {}_aD_t^p f(t) + \mu\, {}_aD_t^p g(t). \end{aligned} $$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Linearity of fractional derivatives

$$ D^p\Big(\lambda f(t) + \mu g(t)\Big) = \lambda\, D^p f(t) + \mu\, D^p g(t) $$

Riemann-Liouville derivatives:

$$ \begin{aligned} {}_a\mathbf{D}_t^p\Big(\lambda f(t) + \mu g(t)\Big) &= \frac{1}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-p-1} \Big(\lambda f(\tau) + \mu g(\tau)\Big) d\tau \\ &= \frac{\lambda}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-p-1} f(\tau) d\tau \\ &\quad + \frac{\mu}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-p-1} g(\tau) d\tau \\ &= \lambda\, {}_a\mathbf{D}_t^p f(t) + \mu\, {}_a\mathbf{D}_t^p g(t). \end{aligned} $$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Left- and right-sided fractional derivatives

$_a\mathbf{D}_t^p f(t)$ — Left derivative

$_t\mathbf{D}_b^p f(t)$ — Right derivative

a — the "past" of $f(t)$ — t — the "future" of $f(t)$ — b

$$_a\mathbf{D}_t^p f(t) = \frac{1}{\Gamma(k-p)} \left(\frac{d}{dt}\right)^k \int_a^t (t-\tau)^{k-p-1} f(\tau) d\tau$$

$$_t\mathbf{D}_b^p f(t) = \frac{1}{\Gamma(k-p)} \left(-\frac{d}{dt}\right)^k \int_t^b (\tau-t)^{k-p-1} f(\tau) d\tau$$

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

Did you notice?

- Definition of fractional derivative involves an integration.
- Integration is a non-local operator (as it is defined on an interval).
- $\Rightarrow$ **Fractional derivative is a non-local operator.**
- $\Rightarrow$ Calculating time-fractional derivative of a function $f(t)$ at some $t = t_1$ requires all the past history, i.e. all $f(t)$ from $t = 0$ to $t = t_1$.
- $\Rightarrow$ Fractional derivatives can be used for modeling systems with memory.
- $\Rightarrow$ Calculating space-fractional derivative of a function $f(x)$ at $x = x_1$ requires all non-local $f(x)$ values.
- $\Rightarrow$ Fractional derivatives can be used for modeling distributed parameter systems.

UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

UPM
UNIVERSITI PUTRA MALAYSIA
AGRICULTURE • INNOVATION • LIFE

BERILMU BERBAKTI
WITH KNOWLEDGE WE SERVE

www.upm.edu.my